Lecture 12 02/25/2019

Maxwell Equations and Electrodynamics (Cont'd)

Green's Function for the Inhomogeneous Wave Equation

In general, the equations for $\vec{A}$ and Φ are inhomogeneous wave equations that have a source term. Using Cartesian coordinates, each of the components of $\vec{A}$, and Φ, satisfies the following equation:

$$\nabla^2 \Psi - \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} = -4\pi\, S(\vec{x},t)$$

The general solution in the unbounded space can be written as:

$$\Psi(\vec{x},t) = \Psi_{hom}(\vec{x},t) + \int S(\vec{x}',t')\, G(\vec{x}-\vec{x}', t-t')\, d^3x'\, dt'$$

Here, Ψ_{hom} is the general solution to the homogeneous wave equation and G is the Green's function satisfying the following equation:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{x}-\vec{x}', t-t') = -4\pi\, \delta^{(3)}(\vec{x}-\vec{x}')\, \delta(t-t')$$

Defining $\vec{R} \equiv \vec{x}-\vec{x}'$ and $\tau \equiv t-t'$, the Green's function obeys:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2}\right) G(\vec{R},\tau) = -4\pi\, \delta^{(3)}(\vec{R})\, \delta(\tau)$$

Due to the isotropy of space, G is a function of $R \equiv |\vec{R}|$, and hence:

$$\nabla^2 G(R,\tau) = \frac{1}{R}\frac{\partial^2}{\partial R^2}\left(R G(R,\tau)\right)$$

Let us write the Fourier transform of G in time domain:

$$G(R,\tau) = \frac{1}{2\pi}\int \underset{\sim}{G}(R,\nu)\, e^{-i\nu\tau}\, d\nu$$

We then have:

$$\left(\nabla^2 + \frac{\nu^2}{c^2}\right)\underset{\sim}{G}(R,\nu) = -4\pi\,\delta^{(3)}(\vec{R})$$

For $R \neq 0$, $\delta^{(3)}(\vec{R}) = 0$. Thus:

$$\frac{1}{R}\frac{\partial^2}{\partial R^2}(R\underset{\sim}{G}) + \frac{\nu^2}{c^2}\underset{\sim}{G} = 0 \Rightarrow \frac{\partial^2}{\partial R^2}(R\underset{\sim}{G}) + \frac{\nu^2}{c^2}(R\underset{\sim}{G}) = 0 \Rightarrow \underset{\sim}{G} = A e^{\frac{i\nu R}{c}} + B e^{-\frac{i\nu R}{c}}$$

In order to satisfy the equation at $R=0$, we must have $\underline{\underline{B = 1 - A}}$.

The Green's function $G(R,\tau)$ is then obtained to be:

$$G(R,\tau) = \frac{A}{2\pi R}\int_{-\infty}^{+\infty} e^{-i\nu\left(\tau - \frac{R}{c}\right)}\, d\nu + \frac{(1-A)}{2\pi R}\int_{-\infty}^{+\infty} e^{-i\nu\left(\tau + \frac{R}{c}\right)}\, d\nu$$

$$\frac{A\,\delta\left(\tau - \frac{R}{c}\right)}{R} + \frac{(1-A)\,\delta\left(\tau + \frac{R}{c}\right)}{R}$$

$$\boxed{G(\vec{x}-\vec{x}', t-t') = A\,\frac{\delta\left(t-t' - \frac{|\vec{x}-\vec{x}'|}{c}\right)}{R} + (1-A)\,\frac{\delta\left(t-t' + \frac{|\vec{x}-\vec{x}'|}{c}\right)}{R}}$$

For $A=1$, we have:

$$G(R,\tau) = G^{(+)}(R,\tau) = \frac{\delta(\tau - \frac{R}{c})}{R}$$

This is called the "retarded" Green's function. It represents the fact that an impulse source at time t' and location $\vec{x}'$ propagates in vacuum isotropically at speed c, and its effect at a location $\vec{x}$ is felt at a later time $\frac{|\vec{x}-\vec{x}'|}{c}$.

The isotropic propagation is clearly seen in the frequency domain where $\underset{\sim}{G}^{(+)}(R,\omega) = \frac{e^{i\frac{\omega}{c}R}}{R}$. This corresponds to a spherically outgoing wave whose amplitude decays $\propto \frac{1}{R}$ with distance.

For $A=0$, we have:

$$G(R,\tau) = G^{(-)}(R,\tau) = \frac{\delta(R + \frac{\tau}{c})}{R}$$

This is called the "advanced" Green's function. It does not have a simple physical interpretation, but it is sometimes useful from a mathematical viewpoint. For $A = \frac{1}{2}$, we have the "symmetric" Green's function $G^{(s)}(R,\tau) = \frac{1}{2}\left[G^{(+)}(R,\tau) + G^{(-)}(R,\tau)\right]$.

The general solution to the inhomogeneous equation is:

$$\psi^{(+)}(\vec{x},t) = \psi^{(+)}_{hom}(\vec{x},t) + \int S(\vec{x}',t')\, G^{(+)}(\vec{x}-\vec{x}', t-t')\, d^3x'\,dt'$$

$$\psi^{(-)}(\vec{x},t) = \psi^{(-)}_{hom}(\vec{x},t) + \int S(\vec{x}',t')\, G^{(-)}(\vec{x}-\vec{x}', t-t')\, d^3x'\,dt'$$

We note that $\lim_{t\to-\infty} G^{(+)}(\vec{x}-\vec{x}', t-t') = 0$ and $\lim_{t\to+\infty} G^{(-)}(\vec{x}-\vec{x}', t-t') = 0$

for all finite t'. Therefore:

$$\lim_{t\to-\infty} \psi^{(+)}(\vec{x},t) = \psi^{(+)}_{hom}(\vec{x},t)$$

$$\lim_{t\to+\infty} \psi^{(-)}(\vec{x},t) = \psi^{(-)}_{hom}(\vec{x},t)$$

The $\psi^{(\pm)}_{hom}(\vec{x},t)$ solutions are known as the in/out solutions. In a scattering problem, $\psi^{(+)}_{in}(\vec{x},t)$ represents the incident wave that is scattered by the source. In an emission problem $\psi^{(+)}_{in} = 0$. Thus:

$$\psi(\vec{x},t) = \int S(\vec{x}',t')\, \frac{\delta\left(t-t'-\frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|}\, d^3x'\,dt' = \int \frac{S\left(\vec{x}', t-\frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|}\, d^3x'$$

One- and Two-Dimesional Green's Functions

A one-dimensional source of the form $\delta(z-z')\,\delta(t-t')$ may be considered as an infinite plane that flashes at $t=t'$. Similarly, a two-dimensional source of the form $\delta^{(2)}(\vec{\rho}-\vec{\rho}\,')\,\delta(t-t')$ is a flashing line at time t'. The nature of one- and two-dimensional Green's functions is quite different from the three-dimensional situation that we have so far discussed.

A simple way of deriving the lower-dimensional Green's functions is to integrate the three-dimensional wave equation over the irrelevant dimensions. For example, consider a one-dimensional source of the form $\delta(z-z')\,\delta(t-t')$. We start with:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{x}-\vec{x}\,', t-t') = -4\pi\,\delta^{(3)}(\vec{x}-\vec{x}\,')\,\delta(t-t')$$

After integrating over $\vec{\rho}$, we find:

$$\frac{\partial^2}{\partial z^2}\int G(\vec{x}-\vec{x}\,', t-t')\,d^2\rho + \underbrace{\int \nabla_T^2 G\,d^2\rho}_{=0} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int G(\vec{x}-\vec{x}\,', t-t')\,d^2\rho$$

$$= -4\pi\,\delta(z-z')\,\delta(t-t')$$

(for a localized source)

Then, defining $G^{(1)}(z-z', t-t') = \int G(\vec{x}-\vec{x}\,', t-t')\,d^2\rho$, we have:

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G^{(1)}(z-z', t-t') = -4\pi\,\delta(z-z')\,\delta(t-t')$$

Note that:

$$G^{(1)}(z-z', t-t') = \int \frac{\delta\left(t-t' - \frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|} d^2\rho \underset{\uparrow}{=} \int \frac{\delta\left(t-t' - \frac{\sqrt{\rho^2+(z-z')^2}}{c}\right)}{\sqrt{\rho^2+(z-z')^2}} d^2\rho$$

shifting $\vec{\rho} \to \vec{\rho} - \vec{\rho}'$

$$= 2\pi \int_0^\infty \frac{\delta\left(t-t' - \frac{\sqrt{\rho^2+(z-z')^2}}{c}\right)}{\sqrt{\rho^2+(z-z')^2}} \rho\, d\rho \underset{\downarrow}{=} 2\pi c \int_{|z-z'|}^\infty \delta\left(t-t' - \frac{R}{c}\right) d\left(\frac{R}{c}\right)$$

$R \equiv \sqrt{\rho^2+(z-z')^2}$

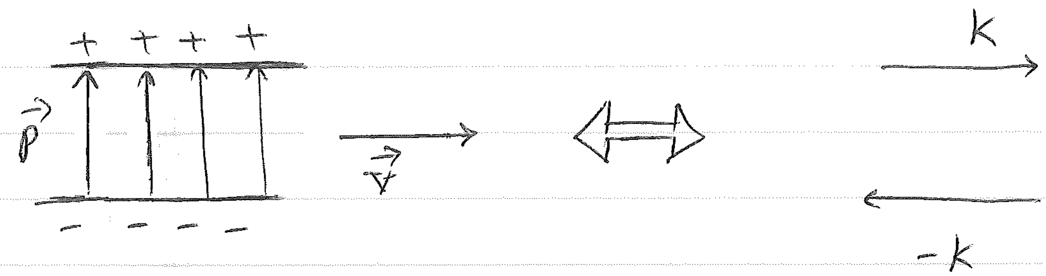
$$\Rightarrow \boxed{G^{(1)}(z-z', t-t') = 2\pi c\, \Theta\left(t-t' - \frac{|z-z'|}{c}\right)}$$

Θ: Heaviside step function

Similarly, we can **show** that $G^{(2)}(\vec{\rho}-\vec{\rho}', t-t') \propto \Theta\left(t-t' - \frac{|\vec{\rho}-\vec{\rho}'|}{c}\right)$.

Moving Media

Motion of a medium carrying charges can give rise to another kind of current called "convective current", which leads to associated magnetization. For example, consider a moving dielectric medium with polarization and bound charges at its boundaries as follows:



The surface current is $\vec{K} = |\vec{P}|\,\vec{v}$ as $\sigma_p = |\vec{P}|$. It is equivalent to a magnetization $\vec{M}$ where:

$$\boxed{\vec{M} = \vec{P} \times \vec{v}} \qquad \left(\vec{K} = \vec{M} \times \hat{n} = (\vec{P} \times \vec{v}) \times \hat{n} = (\vec{P} \cdot \hat{n})\vec{v} - \overset{0}{(\vec{v} \cdot \hat{n})}\vec{P} = |\vec{P}|\,\vec{v} \right)$$

Similarly, a moving magnetization is equivalent to polarization:

$$\boxed{\vec{P} = \frac{\vec{v} \times \vec{M}}{c^2}}$$

An important point is the appearance of c^2 in the denominator in this case, which makes it a <u>purely relativistic</u> effect.